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TESTING FOR PERIODICITY IN A TIME SERIES.(U)

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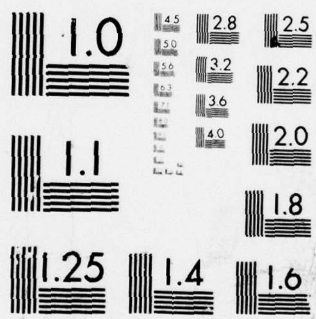

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TESTING FOR PERIODICITY IN A TIME SERIES

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ABSTRACT

In 1929, Sir R. A. Fisher proposed a test for periodicity in a time series based on the maximum spectrogram ordinate. In this paper I propose a one-parameter family of tests that contains Fisher's test as a special case. It is shown how to select from this family a test that will have substantially larger power than Fisher's test against many alternatives, yet will lose only negligible power against alternatives for which Fisher's test is known to be optimal. Critical values are calculated and tabled using a duality with the problem of covering a circle with random arcs. The power is studied using Monte Carlo techniques. The method is applied to the study of the magnitude of a variable star, showing that these power gains can be realized in practice.

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Work Unit Number 4 - Probability, Statistics and Combinatorics.

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† also Technical Report #511, Statistics Department, University of Wisconsin and Technical Report, Statistics Department, Stanford University.

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SIGNIFICANCE AND EXPLANATION

An important problem in the analysis and prediction of physical systems is answering the question "is there periodicity?" when the measurements of its evolution through time contain a random element. Periodicity refers to the regular repetition of a pattern as, for example, with the eleven year cycle of the sun. Problems of this type have often arisen in meteorology, astronomy, and other sciences, and in 1929, Sir R. A. Fisher proposed a statistical test designed for this situation.

In this paper, new statistical procedures are proposed for this problem that represent a generalization of and an improvement upon Fisher's test. The improvement is in the power of these tests: they are more sensitive to the detection of certain types of periodicity without sacrificing overall sensitivity. The procedure is readily applied to data. Instructions and tables as well as theory and examples are included in the report.

TESTING FOR PERIODICITY IN A TIME SERIES[†]

Andrew F. Siegel

Sir R. A. Fisher (1929, 1939, and 1940) proposed a test for periodicity in a time series based on the ratio of the maximum to the sum of the ordinates of the spectrogram or periodogram. In this paper I propose a one-parameter family of tests that contains Fisher's test as a special case. Although Fisher's test is optimal in the case of a simple periodicity, a test can be chosen from this family that loses only negligible power in this case and yet can gain substantial power in the case of compound periodicity. This test is not based just on the largest spectrogram ordinate, but adaptively and continuously on all large values.

Section 2 contains background information, notation, and a review of Fisher's test. The new tests are proposed in section 3 together with a heuristic justification for their consideration. Critical values are calculated and tabled in section 4, having been obtained through a duality discovered by Fisher (1940) and using my recent work in geometrical probability (Siegel, 1978). An example of the use of the tables is also given in section 4. Results of a Monte Carlo power study are presented in section 5, indicating the strengths and weaknesses of these procedures, and providing a method of selecting a good test from this family. Finally, in section 6, the methods are applied to measurements of the magnitude of a variable star in order to show that these potential power gains can be realized in practice.

[†] also Technical Report #511, Statistic Department, University of Wisconsin and Technical Report, Statistics Department, Stanford University.

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2. Background, Notation, and Review of Fisher's Test

Consider a series u_t, $(t=1,\ldots,N)$, observed at equal intervals of time and arising from the model

$$u_t = \zeta_t + \epsilon_t \qquad t = 1,\ldots,N \tag{2.1}$$

where ζ_t represents the unobservable, fixed, "true" value at time t of the phenomenon under study, and ϵ_t is the random error, due to measurement and/or other sources. We will assume independent identical Normal distributions for the errors:

$$\epsilon_t \sim N(0,\sigma^2) \tag{2.2}$$

where σ^2 is unknown. We are interested in statistical inference about the behavior of the sequence ζ_t, particularly regarding periodic activity. The null hypothesis is

$$H_0\colon\ \zeta_1 = \cdots = \zeta_N\ . \tag{2.3}$$

For more background about this model, the reader is referred to section 4.3 of Anderson (1971), to section 5.9 of Bloomfield (1976) and to Fisher (1929, 1939, and 1940).

In this paper, we will consider only frequencies whose periods evenly divide the total series length and we suppose that there is no a priori reason to exclude certain frequencies from consideration. In what follows, we will assume that N is odd and define n by

$$N = 2n + 1\ . \tag{2.4}$$

The method of Fisher (1939) for handling the case of N even may also be used with the methods proposed in this paper.

Define the Fourier coefficients in the usual manner:

$$a_0 = \bar{\zeta} = \frac{1}{N}\sum_{t=1}^{N}\zeta_t \tag{2.5}$$

$$a_j = \sqrt{\frac{2}{N}}\sum_{t=1}^{N}\zeta_t\cos\left(\frac{2\pi jt}{N}\right) \tag{2.6}$$

$$b_j = \sqrt{\frac{2}{N}}\sum_{t=1}^{N}\zeta_t\sin\left(\frac{2\pi jt}{N}\right) \tag{2.7}$$

where $j = 1,\ldots,n$. This uniquely decomposes the sequence of unknown means into periodic components:

$$\zeta_t = a_0 + \sqrt{\frac{2}{N}}\sum_{j=1}^{n}\left[a_j\cos\left(\frac{2\pi jt}{N}\right) + b_j\sin\left(\frac{2\pi jt}{N}\right)\right]. \tag{2.8}$$

The squared amplitude at frequency j/N is

$$R_j^2 = a_j^2 + b_j^2. \tag{2.9}$$

The null hypothesis (2.3) may be equivalently expressed as

$$H_0:\ \text{all } R_j^2 = 0. \tag{2.10}$$

We are interested in all departures from H_0, but of particular interest are the class of alternatives in which there is periodic activity at one frequency only. These will be called simple periodicities and will be denoted

$$H_j:\ R_j^2 > 0,\ \text{all other } R_i^2 = 0. \tag{2.11}$$

Alternatives of periodicity at two or more frequencies will be called compound periodicities. Of particular interest are those representing activity at exactly two frequencies:

$$H_{jk}:\ R_j^2 > 0,\ R_k^2 > 0,\ \text{all other } R_i^2 = 0. \tag{2.12}$$

Estimates $\hat{a}_j$ and $\hat{b}_j$ are obtained by replacing the unobservable ζ_t by the observed series u_t in equations (2.6) and (2.7), and lead to the spectrogram values

$$\hat{R}_j^2 = \hat{a}_j^2 + \hat{b}_j^2. \tag{2.13}$$

To eliminate the effect of σ^2, we normalize these so that they sum to one:

$$Y_j = \hat{R}_j^2 \Big/ \sum_{i=1}^{n}\hat{R}_i^2 \tag{2.14}$$

and we base our inferences on $(Y_1,\ldots,Y_n)$. Fisher (1940) notes that Y_j is the ratio of the sum of squares due to frequency j/N to the total sum of squares; this is because

$$\sum_{i=1}^{n}\hat{R}_i^2 = \sum_{t=1}^{N}(u_t-\bar{u})^2. \tag{2.15}$$

Fisher's test is based on the statistic

$$S = \max_{1\le j\le n} Y_j \tag{2.16}$$

and rejects H_0 when S exceeds the appropriate critical value, g_F. In theorem 4.3.6, section 4.3.4 of Anderson (1971), it is noted that Fisher's test is the uniformly most powerful symmetric invariant decision procedure against simple periodicities.

3. The Procedure

There is no reason to suppose that the optimality property of Fisher's test for simple periodicity extends to compound periodicity, in which there is activity at several frequencies. In this section we give a heuristic argument for why it will not be optimal, and introduce a family of test statistics that should overcome this problem.

Because of the normalization in (2.14), any increase in a smaller Y_j will tend to decrease their maximum, S, and thus lower the power of Fisher's test. This is illustrated in figure 3.1. In the case of simple periodicity only Y_1 gives a large contribution, which exceeds the critical value g_F, and Fisher's test rejects. In the case of compound periodicity, Y_1 and Y_2 are both large, but Y_1 is therefore reduced; now neither exceeds g_F and Fisher's test no longer rejects the null hypothesis.

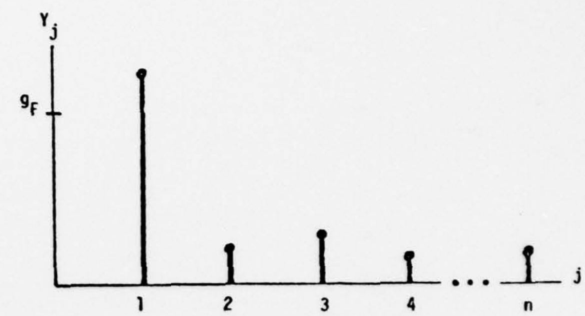
In order to remedy this situation, we should use a test statistic based on all large Y_j, instead of only their maximum. Such a continuous adaptive statistic may be constructed by choosing a threshhold value $g \leq g_F$. For each Y_j that exceeds g, sum the excess of Y_j above g. Setting $\lambda = g/g_F$, the proposed statistic is

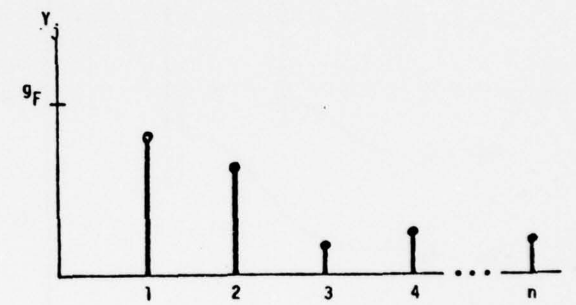
$$T_\lambda = \sum_{j=1}^{n} (Y_j - \lambda g_F)_+ \qquad (3.1)$$

where $(t)_+ = \max(t,0)$ is the positive-part function. H_0 will be rejected when T_λ is large; critical values are found in section 4.

The choice of λ, between 0 and 1, is to be made from theoretical considerations and not from the data itself. $\lambda = 1$ yields Fisher's test

Figure 3.1. A hypothetical spectrogram for simple and compound periodicities.





because $T_1 > 0$ if and only if some Y_j exceeds g_F. A choice of λ near 1 would be used when at most simple periodicities are expected. A smaller value of λ would be used when compound periodicities are a possibility. Further guidance in choosing λ is found in section 5.

A hypothetical spectrogram under H_{12} is shown in figure 3.2. Fisher's test, based on the largest Y_j, does not reject because no Y_j exceeds the critical value g_F. A test based on T_λ may very well reject, because it is based on two large terms, $T_\lambda = (Y_1-\lambda g_F) + (Y_2-\lambda g_F)$, allowing both large Y_j to be counted.

4. Critical Values for T_λ.

The duality discovered by Fisher (1940) between the distribution of the statistic S and the probability of covering a circle with random arcs as treated by Stevens (1939) may be exploited here in order to obtain the distribution of the proposed statistic T_λ under the null hypothesis. My recent work in geometrical probability (Siegel, 1978) leads directly to an exact formula for this distribution, which is presented in this section together with a table of critical values for T_λ and an example of their use.

Fisher's duality is nicely explained in section III.3 of volume II of Feller (1971). The key fact is that $Y_1,\ldots,Y_n$ have the same joint distribution as the lengths of the n gaps produced when n points are independently and uniformly placed on the edge of a circle of circumference one. Figure 4.1 graphically illustrates this geometrical configuration. To make the connection with Stevens' problem, place n arcs of length g, extending counter-clockwise from each of the n random points,

Figure 3.2. A hypothetical spectrogram for comparison of the statistics S and T_λ, $\lambda = .6$, in the case of compound periodicity.

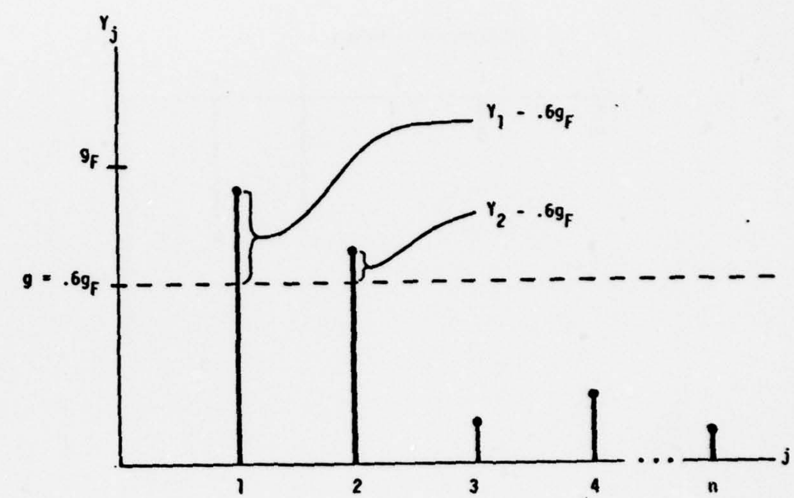
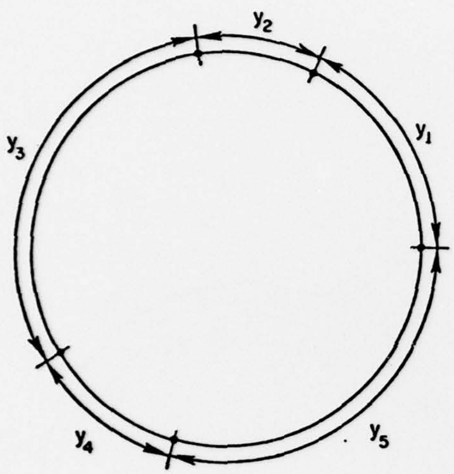


Figure 4.1

Representation of $Y_1,\ldots,Y_n$ as spacings between ordered uniform points on the circle, in the case $n=5$.



as illustrated in figure 4.2. From this one can see, for example, that the probability that no Y_j exceeds g is equal to the probability that n random arcs of length g completely cover the circle.

The corresponding key observation to be made in order to obtain the distribution of T_λ is:

> T_λ has the same distribution as that proportion of the circle that is left uncovered by the union of n random arcs of length $g = \lambda g_F$.

This may be seen from figure 4.2, because $(Y_j-g)_+$ is precisely that proportion of the circle within the gap of length Y_j that is not covered by any arc. In the language of coverage problems, the uncovered proportion is called the vacancy. Its distribution in this case is known (Siegel, 1978) and is given by

$$P_{H_0}(T_\lambda>t) = \sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1} (-1)^{k+\ell+1}\binom{n}{\ell}\binom{\ell-1}{k}\binom{n-1}{k}t^k(1-\lambda\ell g_F-t)_+^{n-k-1} \tag{4.1}$$

Critical values t_λ for T_λ, computed from (4.1), are listed in tables 4.1 through 4.4. These cover significance levels .05 and .01, values of n from 5 through 50, and λ = .2, .4, .6, and .8. If $\lambda = 1.0$, we reject if $T_1 > 0$; this is Fisher's test.

As an example of the use of these tables, suppose we have a time series of length N = 35. Then we use n = 17 because 2n+1 = 35. If we decide to use level .05 and λ = .6, we see from table 4.1 that the initial threshhold is $g = \lambda g_F = .183$. We then compute

Figure 4.2

$Y_1, \ldots, Y_n$ generate n random arcs of length g on the circle, in the case n=5.

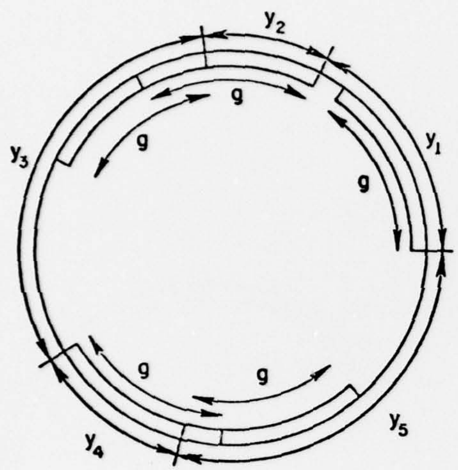


Table 4.1. Level .05 critical values t_λ for T_λ, for several values of λ and for n = 5 through 25.

n	g_F	$.8g_F$	$t_{.8}$	$.6g_F$	$t_{.6}$	$.4g_F$	$t_{.4}$	$.2g_F$	$t_{.2}$
5	.684	.547	.137	.410	.274	.274	.412	.137	.616
6	.616	.493	.123	.370	.246	.246	.381	.123	.587
7	.561	.449	.112	.337	.225	.224	.356	.112	.564
8	.516	.413	.103	.309	.208	.206	.334	.103	.544
9	.477	.382	.0955	.286	.193	.191	.316	.0955	.528
10	.445	.356	.0891	.267	.181	.178	.301	.0890	.513
11	.417	.334	.0835	.250	.171	.167	.287	.0834	.500
12	.392	.314	.0787	.235	.162	.157	.275	.0785	.488
13	.371	.297	.0744	.223	.154	.148	.265	.0742	.478
14	.352	.281	.0707	.211	.147	.141	.255	.0703	.468
15	.335	.268	.0673	.201	.140	.134	.247	.0669	.459
16	.319	.255	.0642	.192	.134	.128	.239	.0638	.451
17	.305	.244	.0615	.183	.129	.122	.232	.0611	.444
18	.293	.234	.0590	.176	.124	.117	.225	.0585	.437
19	.281	.225	.0567	.169	.120	.112	.219	.0562	.430
20	.270	.216	.0546	.162	.116	.108	.213	.0541	.424
21	.261	.208	.0527	.156	.112	.104	.208	.0521	.419
22	.252	.201	.0509	.151	.109	.101	.203	.0503	.413
23	.243	.195	.0492	.146	.106	.0973	.199	.0486	.408
24	.235	.188	.0477	.141	.103	.0941	.195	.0471	.404
25	.228	.182	.0462	.137	.0997	.0912	.190	.0456	.399

Table 4.2. Level .05 critical values t_λ for T_λ, for several values of λ, and for n = 26 through 50.

n	g_F	$.8g_F$	$t_{.8}$	$.6g_F$	$t_{.6}$	$.4g_F$	$t_{.4}$	$.2g_F$	$t_{.2}$
26	.221	.177	.0449	.133	.0971	.0885	.187	.0443	.395
27	.215	.172	.0436	.129	.0946	.0859	.183	.0430	.391
28	.209	.167	.0424	.125	.0923	.0835	.180	.0418	.387
29	.203	.163	.0413	.122	.0901	.0813	.177	.0406	.383
30	.198	.158	.0402	.119	.0880	.0791	.173	.0396	.380
31	.193	.154	.0392	.116	.0861	.0771	.171	.0386	.376
32	.188	.150	.0383	.113	.0842	.0752	.168	.0376	.373
33	.184	.147	.0374	.110	.0824	.0734	.165	.0367	.370
34	.179	.143	.0365	.108	.0807	.0717	.163	.0358	.367
35	.175	.140	.0357	.105	.0791	.0701	.160	.0350	.364
36	.171	.137	.0349	.103	.0776	.0685	.158	.0343	.361
37	.168	.134	.0342	.101	.0761	.0670	.156	.0335	.359
38	.164	.131	.0335	.0984	.0747	.0656	.154	.0328	.356
39	.161	.129	.0328	.0964	.0734	.0643	.151	.0321	.353
40	.157	.126	.0322	.0944	.0721	.0630	.150	.0315	.351
41	.154	.123	.0316	.0926	.0708	.0617	.148	.0309	.349
42	.151	.121	.0310	.0908	.0696	.0605	.146	.0303	.346
43	.148	.119	.0304	.0891	.0685	.0594	.144	.0297	.344
44	.146	.117	.0298	.0874	.0674	.0583	.142	.0292	.342
45	.143	.114	.0293	.0859	.0663	.0572	.141	.0286	.340
46	.141	.112	.0288	.0843	.0653	.0562	.139	.0281	.338
47	.138	.111	.0283	.0829	.0643	.0553	.138	.0276	.336
48	.136	.109	.0279	.0815	.0634	.0543	.136	.0272	.334
49	.134	.107	.0274	.0801	.0625	.0534	.135	.0267	.332
50	.131	.105	.0270	.0788	.0616	.0525	.133	.0263	.330

Table 4.3. Level .01 critical values t_λ for T_λ, for several values of λ, and for n = 5 through 25.

n	g_F	$.8g_F$	$t_{.8}$	$.6g_F$	$t_{.6}$	$.4g_F$	$t_{.4}$	$.2g_F$	$t_{.2}$
5	.789	.631	.158	.473	.315	.315	.473	.158	.642
6	.722	.577	.144	.433	.289	.289	.433	.144	.610
7	.664	.532	.133	.399	.266	.266	.399	.133	.580
8	.615	.492	.123	.369	.246	.246	.372	.123	.555
9	.573	.458	.115	.344	.229	.229	.349	.115	.534
10	.536	.429	.107	.322	.214	.214	.329	.107	.516
11	.504	.403	.101	.302	.202	.201	.313	.101	.500
12	.475	.380	.0950	.285	.190	.190	.298	.0950	.485
13	.450	.360	.0900	.270	.180	.180	.285	.0900	.472
14	.427	.342	.0855	.256	.172	.171	.273	.0854	.461
15	.407	.326	.0814	.244	.164	.163	.262	.0814	.450
16	.389	.311	.0777	.233	.157	.155	.253	.0777	.440
17	.372	.297	.0744	.223	.150	.149	.244	.0744	.431
18	.357	.285	.0714	.214	.144	.143	.236	.0713	.423
19	.343	.274	.0686	.206	.139	.137	.229	.0685	.415
20	.330	.264	.0660	.198	.134	.132	.222	.0659	.408
21	.318	.254	.0636	.191	.129	.127	.215	.0636	.401
22	.307	.245	.0614	.184	.125	.123	.210	.0614	.395
23	.297	.237	.0594	.178	.121	.119	.204	.0593	.389
24	.287	.230	.0575	.172	.117	.115	.199	.0574	.383
25	.278	.223	.0557	.167	.114	.111	.194	.0556	.378

Table 4.4. Level .01 critical values t_λ for T_λ, for several values of λ, and for n = 26 through 50.

n	g_F	$.8g_F$	$t_{.8}$	$.6g_F$	$t_{.6}$	$.4g_F$	$t_{.4}$	$.2g_F$	$t_{.2}$
26	.270	.216	.0541	.162	.110	.108	.190	.0540	.373
27	.262	.210	.0525	.157	.107	.105	.185	.0524	.368
28	.255	.204	.0511	.153	.105	.102	.181	.0509	.364
29	.248	.198	.0497	.149	.102	.0991	.178	.0496	.359
30	.241	.193	.0484	.145	.0993	.0965	.174	.0483	.355
31	.235	.188	.0471	.141	.0969	.0940	.171	.0470	.351
32	.229	.183	.0460	.138	.0946	.0917	.167	.0458	.348
33	.224	.179	.0449	.134	.0925	.0895	.164	.0447	.344
34	.218	.175	.0438	.131	.0904	.0874	.161	.0437	.341
35	.213	.171	.0428	.128	.0884	.0854	.159	.0427	.337
36	.209	.167	.0419	.125	.0866	.0834	.156	.0417	.334
37	.204	.163	.0410	.122	.0848	.0816	.153	.0408	.331
38	.200	.160	.0401	.120	.0831	.0799	.151	.0399	.328
39	.196	.156	.0393	.117	.0815	.0782	.148	.0391	.325
40	.192	.153	.0385	.115	.0799	.0766	.146	.0383	.322
41	.188	.150	.0377	.113	.0784	.0751	.144	.0376	.320
42	.184	.147	.0370	.110	.0770	.0736	.142	.0368	.317
43	.181	.144	.0363	.108	.0756	.0722	.140	.0361	.315
44	.177	.142	.0356	.106	.0743	.0709	.138	.0355	.312
45	.174	.139	.0350	.104	.0730	.0696	.136	.0348	.310
46	.171	.137	.0343	.103	.0718	.0684	.134	.0342	.308
47	.168	.134	.0338	.101	.0706	.0672	.133	.0336	.306
48	.165	.132	.0332	.0990	.0695	.0660	.131	.0330	.303
49	.162	.130	.0326	.0973	.0684	.0649	.129	.0324	.301
50	.160	.128	.0321	.0957	.0673	.0638	.128	.0319	.299

$$T_{.6} = \sum_{j=1}^{17} (Y_j - .183)_+ \qquad (4.2)$$

which includes only those terms for which $Y_j > .183$. $T_{.6}$ is then compared to the critical value $t_{.6} = .129$, also found in table 4.1. If $T_{.6} > .129$, then we reject the null hypothesis.

5. Power Study of Tests Based on T_λ.

The heuristic arguments of section 3 suggest that the tests based on T_λ will be more powerful than Fisher's test against alternatives of compound periodicity. The results of a Monte Carlo power study are now presented that not only confirm this, but also yield two further dividends. First, when Fisher's test is optimal, we find only a negligible loss of power when using T_λ instead, over a wide range of values of λ. Second, the graphs of this section suggest a good choice of λ to use in practice.

When the null hypothesis fails to hold, the spectrogram ordinates $\hat{R}_j^2$ are, up to scale, independently distributed as noncentral Chi-Squares with two degrees of freedom and noncentrality parameters R_j^2, the squared amplitudes at the frequencies j/N (j=1,...,n). Using the computer, the proper pseudorandom noncentral Chi-Squares were generated. From these the statistics T_λ were calculated, and it was noted whether each test rejected or not. Each power estimate is based on 10,000 repetitions, and thus has a standard deviation of less than .005, as calculated for the binomial distribution. Computations were done on Stanford University's IBM 370 and on the University of Wisconsin's Univac 1110 computers, using the pseudorandom number generators RANDK and RANUN respectively.

The results are presented graphically, for significance levels .05 and .01 at n = 10 and 25. Each curve is a graph of the power of T_λ

as a function of λ in the case as labelled. Note that the power of Fisher's test is the height of the extreme right of each curve, corresponding to $\lambda = 1$. The presentation is simplified because the power remains fixed when the amplitudes R_j are permuted among the frequencies j/N. Thus power is a function of the significance level, the values of n and λ, and a list of amplitudes. The actual assignment of amplitudes to frequencies need not be specified.

The case of simple periodicity is shown in figure 5.1 for various amplitudes of periodic activity at one frequency only. Fisher's test is optimal in this case, as noted in section 2, and indeed the curves do slope downwards to the left, illustrating loss of power as we depart from $\lambda = 1$. Note, however, that the curves are nearly horizontal over the range $.6 < \lambda < 1.0$, indicating practically no loss of power in this range if we use T_λ instead of Fisher's test. In fact, only a small amount of power is lost for λ as low as .4; substantial power loss begins for λ in the range .2 to .4. Of course, we don't want to choose λ too close to zero because T_0 is identically one, and data-independent tests are generally frowned upon.

Several cases of compound periodicity are considered. Power in the case of equal amplitudes at each of two frequencies is illustrated in figure 5.2. The fact that these curves now slope upwards to the left (when $.4 < \lambda < 1.0$) indicates that one gains substantial power in these cases by departing from Fisher's test and choosing λ smaller than one. These gains continue down to $\lambda = .4$, after which there eventually must be a loss of power.

Figure 5.1. Estimated power of T_λ as a function of λ under simple periodicity of amplitude R_1, indicated next to curve.

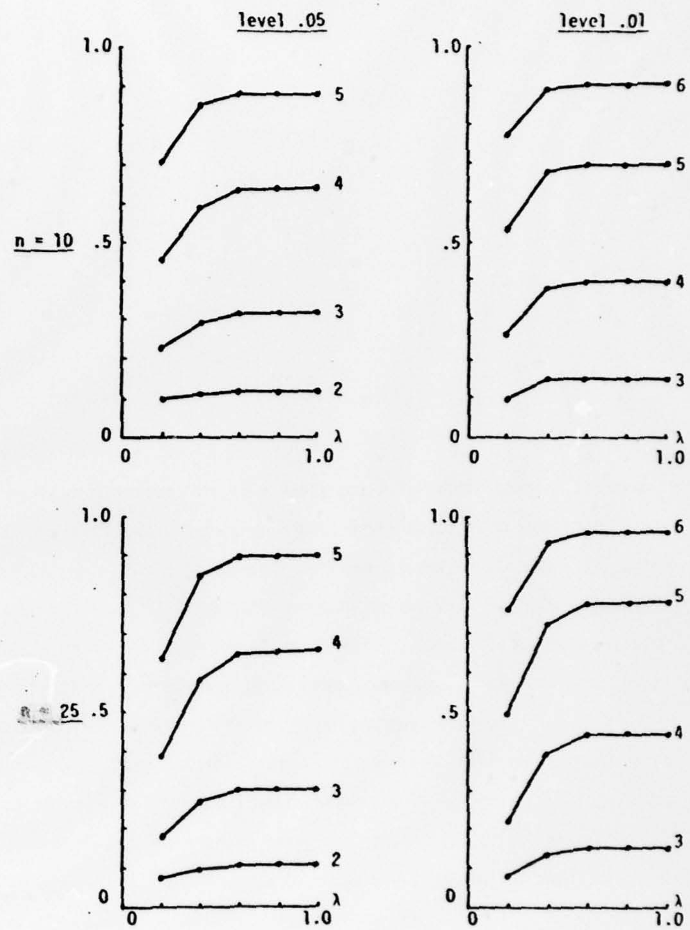
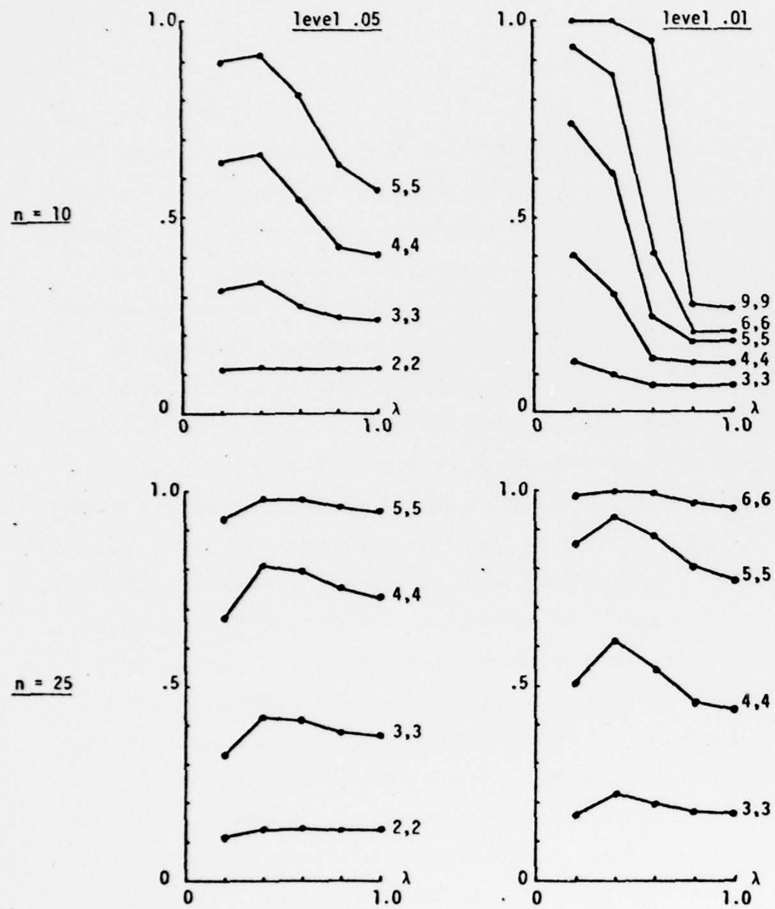


Figure 5.2. Estimated power of T_λ as a function of λ under compound periodicity at two frequencies with equal amplitudes R_1 and R_2, indicated next to curve.



Power in the case of unequal amplitudes at each of two frequencies is illustrated in figure 5.3; one amplitude is twice the other. Again the curves generally slope upwards to the left (when $.6 < \lambda < 1.0$), but the power increases available are less dramatic here than they were in the case of equal amplitudes (figure 5.2).

Power for contributions at three frequencies is illustrated in figure 5.4 for the case of equal amplitudes, and in figure 5.5 for the case of unequal amplitudes having the proportions 1:2:3. We see again that power generally increases as λ decreases from 1.0 to .4, sometimes dramatically, as in figure 5.4 when $n = 10$.

The main conclusion to be drawn from this section is that substantial power gains are often available by using T_λ instead of Fisher's test, without sacrificing significant power in the case of simple periodicity when Fisher's test is optimal. A conservative choice for λ is .6; a choice of $\lambda = .4$ often allows even larger power gains under compound periodicity at the cost of a small but significant power loss under simple periodicity.

Figure 5.3. Estimated power of T_λ as a function of λ under compound periodicity at two frequencies with unequal amplitudes R_1 and R_2, indicated next to curve.

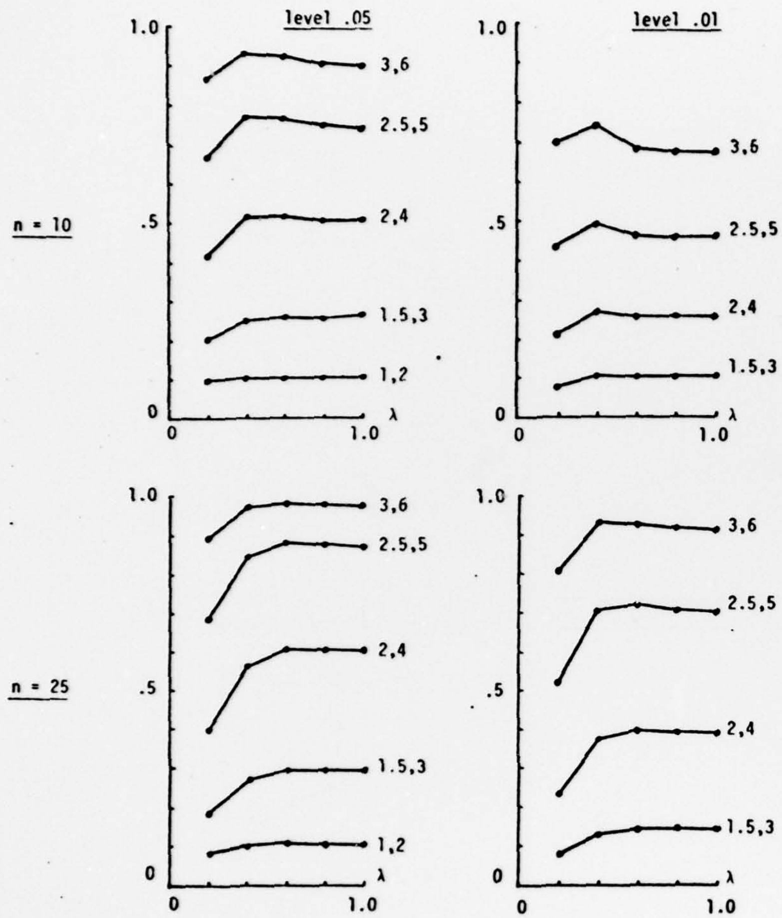


Figure 5.4. Estimated power of T_λ as a function of λ under compound periodicity at three frequencies with equal amplitudes R_1, R_2, and R_3, indicated next to curve.

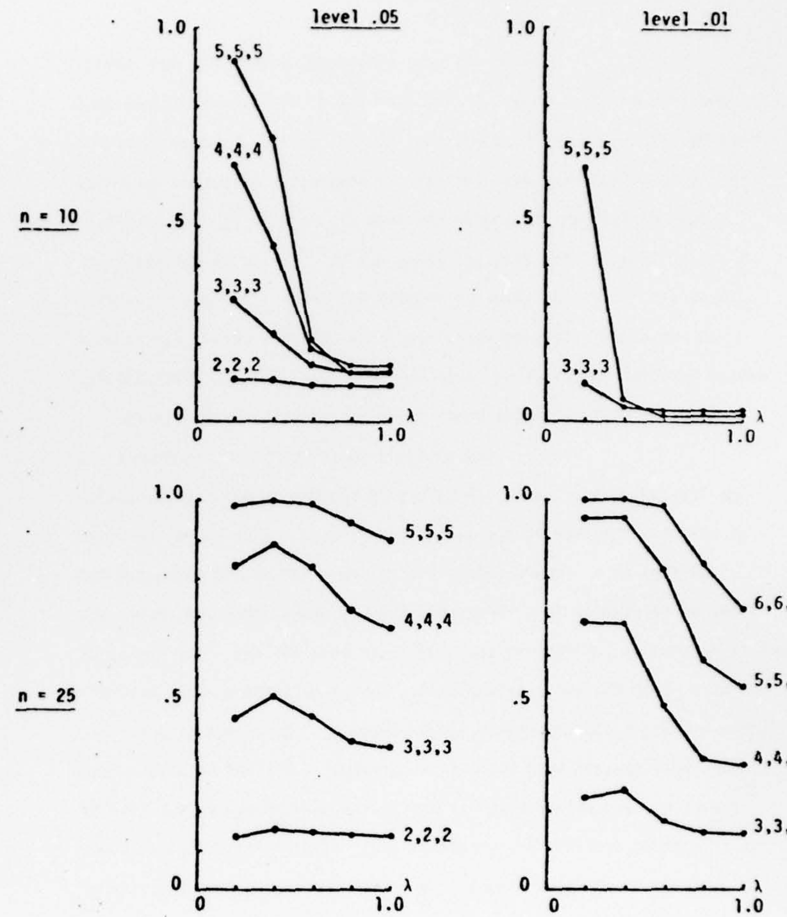
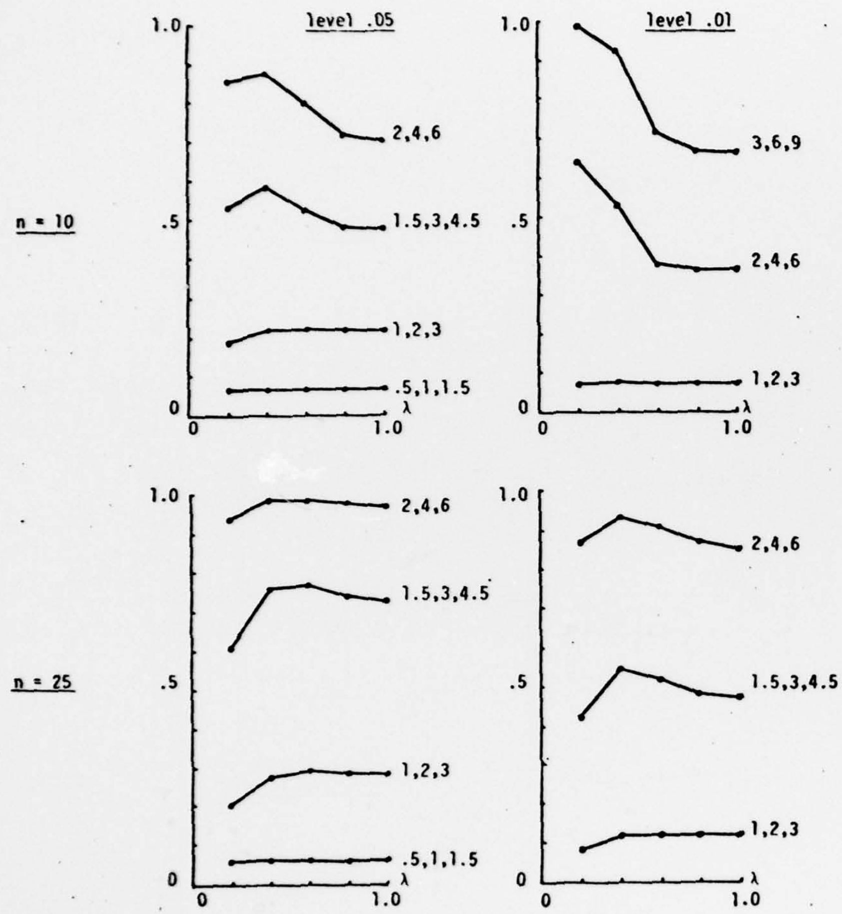


Figure 5.5. Estimated power of T_λ as a function of λ under compound periodicity at three frequencies with unequal amplitudes R_1, R_2, and R_3, indicated next to curve.



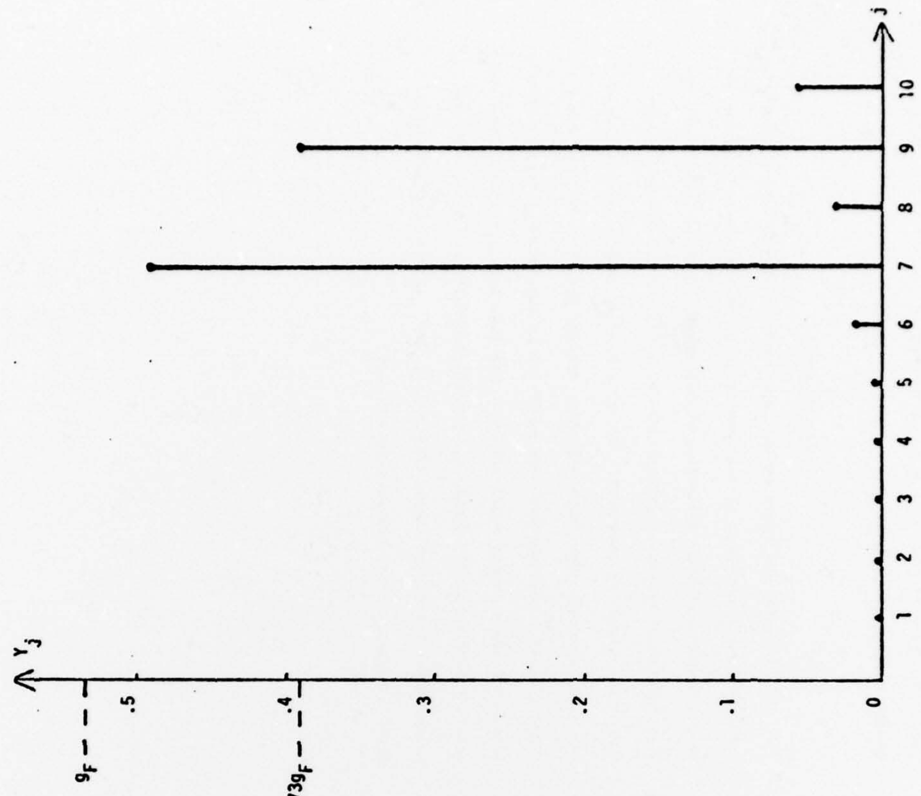
6. Application: Variable Star Data

In order to demonstrate that these potential power gains can be realized in real data situations, we now apply them to the analysis of the magnitude of a variable star. The data is taken from pages 349-352 of Whittaker and Robinson (1924), and has been analyzed in chapters 2 and 5 of Bloomfield (1976). This example is appropriate because it is an essentially closed physical system in which periodicity is likely, and we have no auxiliary information favoring some periods over others.

We will analyze $N = 21$ measurements of the magnitude (thus $n = 10$), obtained from observation at ten day intervals. The raw data is shown in figure 6.1. The spectrum was calculated as outlined in section 2, and the normalized spectrogram is shown in figure 6.2, normalized so that the ordinates sum to one. We see two strong peaks, at periods of about 30 and 23 days. This is not surprising because the raw data in figure 6.1 do seem to exhibit a pattern of "beats" characteristic of the superposition of two close frequencies.

We wish to test to see if these peaks represent true periodic fluctuations in the magnitude of the star, or if they might have arisen from purely random fluctuations. Tests for periodicity may now be compared. Table 6.1 shows the outcome of level .01 tests; all level .05 tests did reject H_0. In the level .01 case, we see that Fisher's test (based on T_λ with $\lambda = 1$) does not reject H_0, largely for the arguments presented in section 3. However, the tests based on T_λ do reject H_0 when $\lambda = .6$, .4, and .2, and accept H_0 when $\lambda = .8$. Recall from section 5 that $\lambda = .6$ and possibly $\lambda = .4$ were the recommended values, and these were not chosen from the data!

Figure 6.2. Normalized spectrogram for variable star data



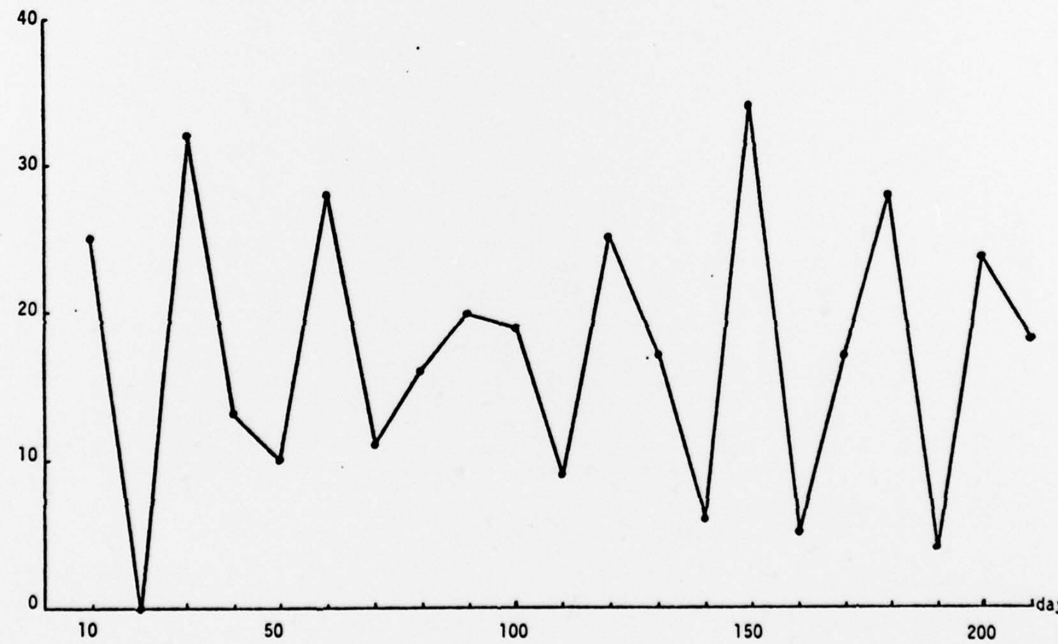


Figure 6.1. Variable star data: brightness sampled at ten-day intervals

Table 6.1. A comparison of tests for periodicity in the variable star data, at level .01. $\lambda = 1.0$ corresponds to Fisher's test.

λ	λg_F	T_λ	t_λ	reject H_0?
1.0	.536	0	0	no
.8	.429	.065	.107	no
.6	.322	.242	.214	yes
.4	.214	.457	.329	yes
.2	.107	.671	.516	yes

If we consider the daily observations (600 instead of 21 points) as analyzed in Bloomfield, we see that there really is periodicity, and hence we do hope to reject the null hypothesis. Thus the extra power gained by using T_λ with $\lambda = .6$ or .4 instead of Fisher's test can be quite useful in practice.

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In 1929, Sir R. A. Fisher proposed a test for periodicity in a time series based on the maximum spectrogram ordinate. ~~In~~ this paper ~~I~~ propose a one-parameter family of tests that contains Fisher's test as a special case. It is shown how to select from this family a test that will have substantially larger power than Fisher's test against many alternatives, yet will lose only negligible power against alternatives for which Fisher's test is known to be optimal. Critical values are calculated and tabled using a duality with the problem of covering a circle with random arcs. The power is studied using Monte Carlo techniques. The method is applied to the study of the magnitude of a variable star, showing that these power gains can be realized in practice.

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